Resonant exit time in stochastic and deterministic systems

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(Received 6 May 2004; revised manuscript received 16 November 2004; published 21 June 2005)

The motion of a particle in the field of a time-dependent potential is studied here both at absolute zero and in the presence of thermal agitation. The potential executes either random fluctuations or deterministic harmonic oscillations. Assuming absorbing boundaries it is always possible to find an exit time $\tau_{ex}(\kappa)$ which has a local minimum as a function of the potential flip rate κ. Thus resonant activation, usually associated with diffussive systems, exists in purely deterministic systems as well. Thermal agitation merely extends the range of admissible initial conditions and renders all exit times finite.

DOI: 10.1103/PhysRevE.71.067102 PACS number(s): 02.50.Ey, 02.60.Cb, 05.10.Gg

I. INTRODUCTION

Resonant activation was initially described in overdamped, thermally activated metastable systems confined by a potential whose barrier height executes random dichotomic Markovian fluctuations [1,2]. The mean first passage time τ out of the metastable region has in this case a local minimum as a function of the fluctuation rate κ. More recently, the phenomenon was studied in some other systems with a fluctuating barrier [3,4] and was shown [5] to exist also in overdamped systems whose potential barrier height executes deterministic oscillations.

In this work we investigate the exit problem of an inertial Brownian particle confined by a randomly fluctuating potential. We find that in full agreement with the Kramers theory of thermal activation [6] the κ-dependent mean exit time depends in a nonmonotonic fashion on the dissipation strength. However, we then also show that a very similar phenomenon exists in nondiffusive, deterministic systems as well. The thermally driven diffusive process does not change the qualitative properties of the resonance effect, but merely extends the range of admissible initial conditions and renders all exit times finite.

The motion of an inertial particle with unit mass is described here by the Langevin equation,

$$\ddot{x} + \eta\dot{x} + V'(x,t) = \sqrt{2\eta T}w(t), \tag{1}$$

where $\dot{x}=dx/dt$, η is the dissipation constant, $V(x,t)$ is a time-dependent potential to be defined below, $V'=dV/dx$, $T \geqslant 0$ is temperature ($k_B=1$), and $w(t)$ is the standard white-noise process with two-time correlation $\langle w(t_1)w(t_2)\rangle=\delta(t_1-t_2)$. The equation of motion (1) is solved using the power-series expansion

$$x(t+\Delta) = x(t) + v(t)\Delta + \sum_{i=2}^{9} A_i\Delta^i, \tag{2}$$

with $v(t)=\dot{x}(t)$ and the $A_i=A_i[x(t),v(t)]$. The white-noise terms $w(t)$, where required, are generated by the Box-Mueller algorithm [7].

For definiteness of discussion we shall assume that Eq. (1) describes deterministic motion if $T=0$ and stochastic (random) motion if $T>0$.

II. A THERMALLY ACTIVATED SYSTEM

The exit problem of a thermally activated inertial particle is studied here for the simple case of a randomly switching harmonic potential

$$V(x,t) = \pm V_0x^2/2, \tag{3}$$

$V_0>0$. The random waiting time between successive flips of the potential is ϑ, and we assume that ϑ has the Markovian distribution

$$P(\vartheta) = \kappa e^{-\kappa\vartheta}, \tag{4}$$

where κ is the mean flip rate. The data presented below are based on $N=3500$ realizations of the random process $\{x(t),\dot{x}(t)\}$. Absorbing boundaries are assumed at the points $x=\pm L$, and the random values $\tau^{(i)}$ of the first passage time are calculated for the N realization of the stochastic process $x^{(i)}(t)$, $i\in\langle 1,N\rangle$. The computed mean value

$$\tau_{ex}(N) = \frac{1}{N}\sum_{i=1}^{N}\tau^{(i)}$$

appears to converge to its limit $\tau_{ex}(\infty)$ as [8]

$$|\tau_{ex}(N) - \tau_{ex}(\infty)| \propto \tau_{ex}(\infty)N^{-1/2}.$$

Finally, the random values of the exit velocity are defined as $v^{(i)}=\dot{x}(\tau^{(i)})$.

The exit properties are apparent from Fig. 1 where we plot the mean first passage time $\tau_{ex}=\tau_{ex}(\kappa)$ out of the interval $x\in(-L,L)$, $L=1$, at selected values of the dissipation constant η. In the simulations we set $x(0)=0$ and $v(0)=0$, and in order to maximize the resonance effect (see Fig. 4 below and the discussion of Ref. [5]) we also set $V(x,0)\geqslant 0$ throughout.

In a nonfluctuating potential the mean first passage time has a local minimum [6] as a function of the dissipation strength η. According to Fig. 1 a corresponding effect exists in the fluctuating potential at all values of the flip rate κ (the local minimum is here close to $\eta=1$). In the limit $\kappa\to\infty$ the mean first passage time approaches the free-particle result [2], while in the limit $\kappa\to 0$ the particle most probably exists shortly after the first flip of the potential $V(x,t)$, and $\tau_{ex} \gtrsim \kappa^{-1}$ as shown in the plot. With decreasing temperature T the minimum of $\tau_{ex}(\kappa)$ shifts to the left for all values of η.

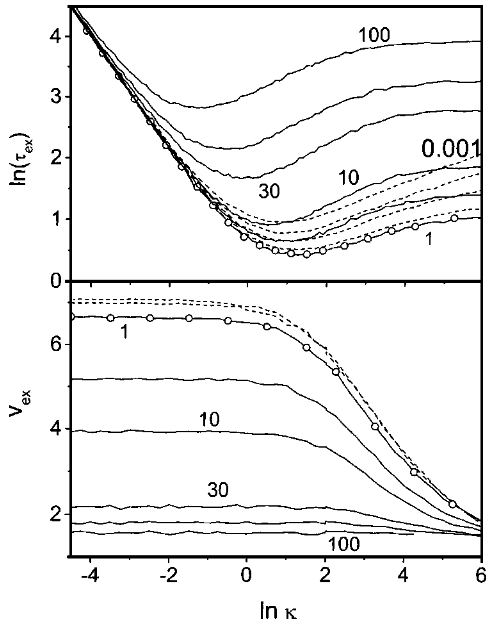

FIG. 1. The mean first passage time τ_{ex} (top) out of the interval $(-1,1)$ and the corresponding mean exit velocity v_{ex} (bottom) vs the mean flip rate κ. Temperature $T=1$, and the potential amplitude $V_0=50$. Initial conditions $x(0)=0$, $v(0)=0$, and $V(x,0)\geqslant 0$. Dissipative strength $\eta=0.001$ (labeled), 0.01, 0.05, 0.25, 1 (labeled, ○-marked), 5, 10 (labeled), 30 (labeled), 50, and 100 (labeled). The $\eta<1$ curves are shown in dashed lines. In the bottom plot the $\eta=0.001$ and 0.01 curves are too close to the other two small η plots and are therefore omitted.

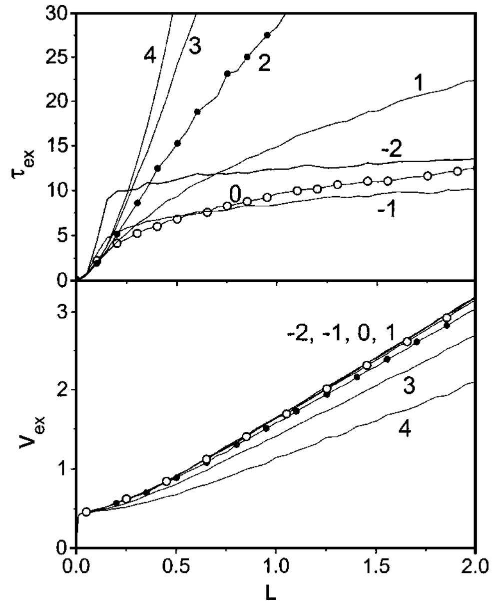

FIG. 2. The mean first passage time τ_{ex} (top) out of the interval $(-L,L)$ and the corresponding mean exit velocity v_{ex} (bottom) vs the interval width L. Temperature $T=0.1$ and the potential amplitude $V_0=50$. Initial conditions as in Fig. 1, dissipation strength $\eta=30$. The fluctuation rate κ is $\ln\kappa=-2$, -1, 0, 1, 2, 3, and 4 as labeled.

In the nonfluctuating potential the mean exit velocity $v_{ex}=v_{ex}(\eta)$ is all but constant at small η, and decreases to zero as $\eta\to\infty$. These two asymptotic regions are separated by an indistinct local maximum which is located a little to the right of the minimum of $\tau_{ex}(\eta)$; these two regions are easily discerned in the plots of Fig. 1. Similar to $\tau_{ex}(\kappa)$, the function $v_{ex}(\kappa)$ approaches the free-particle values as $\kappa\to\infty$. Simulations of the underdamped free-particle motion suggest that

$$\tau_{ex}=(6\eta TL^{-2})^{-1/3}, \tag{5}$$

$$v_{ex}=(6\eta TL)^{1/3}. \tag{6}$$

The fitted Eq. (5) is comparable to the analytic results of Ref. [9]. In the overdamped limit one has further the standard result [6],

$$\tau_{ex}=\eta T^{-1}L/2, \tag{7}$$

while at sufficently long exit times $\tau_{ex}\gg\eta^{-1}$ there is

$$v_{ex}=T^{-1/2}. \tag{8}$$

The dependence of the two functions $\tau_{ex}=\tau_{ex}(\eta,\kappa|L)$ and $v_{ex}=v_{ex}(\eta,\kappa|L)$ on the interval width L is determined by the detailed form of the potential $V(x,t)$. Representative plots of the two functions $\tau_{ex}=\tau_{ex}(L)$ and $v_{ex}=v_{ex}(L)$ for the harmonic potential of Eq. (3) are shown in Fig. 2. The plots show that at small values of the fluctuation rate κ the function $\tau_{ex}=\tau_{ex}(L)$ increases initially very rapidly with small L, and then only very slowly with large L. We assume this slowdown to be an inertial effect which vanishes at large fluctuation rates where the function $\tau_{ex}=\tau_{ex}(L)$ increases rapidly at all values of L. With decreasing dissipation strength η this inertial behavior extends progressively to higher values of the fluctuation rate κ.

The exit velocity $v_{ex}=v_{ex}(L)$ has an initial rapid step followed by gradual growth at larger values of L. The magnitude of the initial step decreases with decreasing values of η and T, and we assume that the step represents the initial (partial) equilibrization of the particle with the ambient heat bath. Interestingly, at low temperatures and low damping there is $v_{ex}(L)\propto L$ for almost all values of L. This width dependence seems to be an artifact of the harmonic motion.

III. SYSTEMS AT $T=0$

In this section we show that the exit time becomes a nonmonotonic function of the flip rate κ even in the absence of thermally driven diffusion. The phenomenon, moreover, exists both in systems executing random fluctuations of the potential $V=V(x,t)$ and in systems executing deterministic oscillations.

We consider first the piecewise deterministic motion in the field of the randomly switching harmonic potential (3). The same as before, we assume $V(x,0)\geqslant 0$, but in order to obtain nontrivial solutions of the homogeneous, $T=0$, equation (1) we now impose the nontrivial initial conditions $x(0)=0$ and $v(0)=\pm 1$ with equal probability. The resultant mean first passage time averaged over the random flips of the potential (3) is plotted in Fig. 3. In the limit $\kappa\to 0$ there is

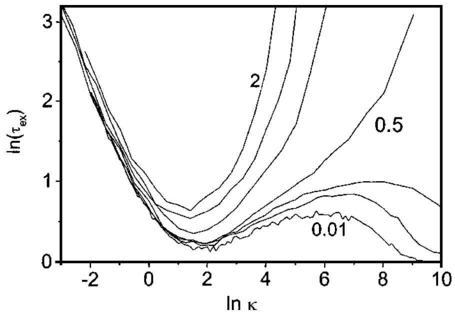

FIG. 3. The mean first passage time τ_{ex} out of the interval $(-1,1)$ vs the mean flip rate κ. Temperature $T=0$, and the potential amplitude $V_0=50$. Initial conditions $x(0)=0$, $v(0)=\pm 1$ with equal probability, and $V(x,0)\geqslant 0$. Dissipative strength $\eta=0.01$ (labeled), 0.2, 0.3, 0.5 (labeled), 1, 1.5, and 2 (labeled). Unsmoothed data curves.

$\tau_{ex}\gtrsim \kappa^{-1}$ as before, but the large κ behavior depends on the dissipation strength η: At sufficiently large values of κ and η the particle becomes trapped in a quasiperiodic trajectory entirely enclosed within the interval $x\in(-1,1)$ and the exit time is in this case infinite (see, e.g., the $\eta=2$ curve of Fig. 3). At small η and large κ, on the other hand, the particle exits as if it were essentially free, as shown here by the $\eta=0.01$ curve. With increasing initial velocity this behavior extends to ever larger values of the dissipation strength.

A family of curves very similar to those shown in Fig. 3 can also be obtained by setting $\eta=0$ and by varying the initial energy of the particle. In this case, apparently, all exit times are finite.

In general we find that the resonance effect in piecewise deterministic systems exists only in a narrow range of parameters. For example, if the potential $V(x,t)$ oscillates between the values $V(x)=V_0x^2/2$ and $V(x)=0$, then the exit time $\tau_{ex}(\kappa)$ has a local minimum at $\kappa\approx 1$ if $V(x,0)\geqslant 0$, $V_0=10$, and $L\lesssim 0.6$ provided that $x(0)=0$ and the values of $v(0)$ and η are not too high. Interestingly, in this case the exit time is infinite at both very small and at very large values of the fluctuation rate κ.

The above simple examples demonstrate that there exists no qualitative difference between the exit properties described by the $T=0$, homogeneous equation of motion (1), and by the $T>0$, inhomogeneous equation of motion. The inhomogeneous term merely allows for nontrivial solutions with the trivial initial conditions $x(0)=0$ and $v(0)=0$. Under the influence of this term, moreover, the Brownian particle may cross the separatrix, so that all exit times become finite, in contrast to the $T=0$ case where infinite exit times are possible.

We wish now to conclude the paper by demonstrating that the resonant effect exists also in purely deterministic systems with oscillating barriers, and to this end we set

$$V(x,t) = V_0\frac{x^2}{2}\cos(\kappa t+\phi) \qquad (9)$$

in Eq. (1). The $T>0$ overdamped dynamics of this system were analyzed in Ref. [5] where it was shown that the harmonically oscillating potential leads to a resonance effect fully comparable to the customary resonance [1] due to random fluctuations of the barrier height. The deterministic exit times of an inertial particle at $T=0$ are shown in Fig. 4. The resonance effect exists here at sufficiently small values of the initial velocity $v(0)\neq 0$, and at a suitable choice of the phase ϕ. This phase dependence was discussed in Ref. [5]. A particle with a high initial velocity again exits as if it were essentially free.

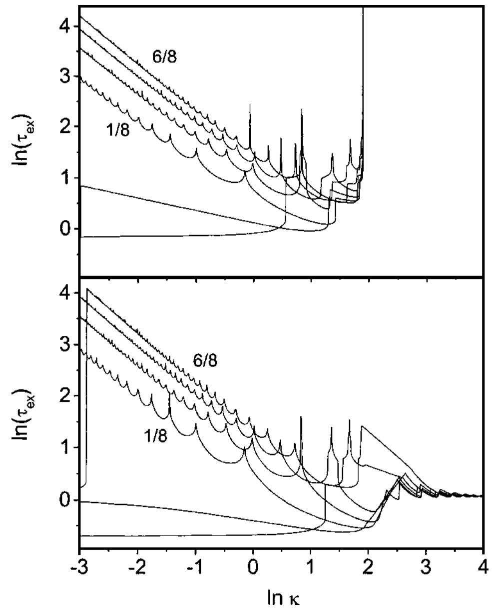

FIG. 4. The exit time τ_{ex} out of the interval $(-1,1)$ vs the oscillatory rate κ for the inertial deterministic motion given at $T=0$ by Eqs. (1) and (5). The phase $\phi/2\pi=4/8$ (lowermost curves), 2/8, 1/8 (labeled), 0, 7/8, and 6/8 (topmost curves, labeled). Initial conditions $x(0)=0$ and $v(0)=0.02$ (top), and $x(0)=0$ and $v(0)=1$ (bottom). Dissipation strength $\eta=0.1$.

In the overdamped limit the above equation of motion becomes

$$\dot{x} = -V_0 x\cos(\kappa t+\phi). \qquad (10)$$

The explicit solution of this equation,

$$x(t) = x(0)\exp\{V_0\kappa^{-1}[\sin\phi - \sin(\kappa t+\phi)]\}, \qquad (11)$$

yields exit times which do not qualitatively differ from the $\tau_{ex}(\kappa,\phi)$ curves shown in Fig. 4 (top).

In summary, therefore, we conclude that the origin of resonant activation is to be found in the exit behavior of a deterministic particle confined by a potential with time-dependent barrier height. Inclusion of thermal activation extends the range of the resonant effect to trajectories which originate in the stable points of the potential, and it also renders all exit times finite.

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